Large sets of Steiner systems and an Erdős–Rosenfeld colouring problem: the cases $k = 10$ and $k = 12$

Pikainu*

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Abstract

Erdős and Rosenfeld asked whether there is a $k > 2$ for which the k-subsets of $\{1, \dots, 2k\}$ can be coloured with $k+1$ colours so that every $(k+1)$-subset has all $k+1$ colours among its k-subsets; equivalently, whether $\chi(J(2k, k)) = k + 1$ for some $k > 2$ (Erdős problem 835). The answer was known to be negative for $3 \leq k \leq 8$ and, by a theorem of Ma and Tang, for every k with $k+1$ not prime; the smallest open case was $k = 10$.

We observe that such a colouring exists if and only if the k-subsets of $[2k]$ admit a partition into $k+1$ pairwise disjoint Steiner systems $S(k-1, k, 2k)$, that is, a large set $\mathrm{LS}[k+1](k-1, k, 2k)$. Three consequences follow.

First, iterating the derived-design construction, the existence of $S(k-1, k, 2k)$ forces the existence of the Steiner quintuple system $S(4, 5, k+5)$. Since $S(4, 5, 15)$ and $S(4, 5, 17)$ do not exist, the Erdős–Rosenfeld question has a negative answer for $k = 10$ and for $k = 12$. The smallest open case is now $k = 16$, which requires $S(4, 5, 21)$ — the smallest Steiner quintuple system whose existence is undecided.

Second, the divisibility conditions necessary for $S(k-1, k, 2k)$ hold if and only if $k+1$ is prime, which gives a short elementary reproof of the Ma–Tang theorem.

Third, we develop the structure theory of the tight systems $S(k-1, k, 2k)$. Every such system automatically meets every $(k+1)$-set in exactly one block, so these are precisely the independent sets of $J(2k, k)$ attaining the ratio bound; for even k each is self-complementary; and any two disjoint ones have a completely determined intersection distribution, seen identically from every block. The integrality of those local intersection numbers holds exactly when $k+1$ is prime, so the Ma–Tang condition resurfaces a second time. Exhaustive computation shows that the disjointness graph of tight systems is triangle-free for $k = 4$ and $k = 6$: at most two are pairwise disjoint, whereas a large set needs $k+1$. We conjecture this for all $k \geq 4$, which would settle the Erdős–Rosenfeld question completely and, unlike the derived-design argument, does not require knowing whether a single tight system exists.

1 Introduction

For $0 \leq k \leq n$ the *Johnson graph* $J(n, k)$ has as vertices the k-subsets of $[n] = \{1, \dots, n\}$, two of them adjacent when they meet in exactly $k-1$ points. The following question of Erdős and Rosenfeld is problem 835 in the Erdős problem database [1].

Question 1. Is there a $k > 2$ such that the k-subsets of $\{1, \dots, 2k\}$ can be coloured with $k+1$ colours in such a way that for every $A \subseteq \{1, \dots, 2k\}$ with $|A| = k+1$ all $k+1$ colours occur among the k-subsets of A?

*Correspondence: `mrtt0817@gmail.com`. The mathematical and computational work reported here was carried out with Claude (Anthropic); see the Acknowledgements.

The k-subsets of such an A are pairwise adjacent in $J(2k,k)$ and there are $k+1$ of them, so Question 1 asks exactly whether $\chi(J(2k,k)) = k+1$, the clique number of $J(2k,k)$ being $k+1$ (Lemma 5). The answer is positive for $k=2$ and, as recorded in [1], negative for $3 \le k \le 8$; moreover Ma and Tang proved $\chi(J(2k,k)) > k+1$ whenever $k > 2$ and $k+1$ is not prime. The smallest open case was therefore $k = 10$.

Recall that a *Steiner system* $S(t,\kappa,v)$ is a family of κ-subsets (*blocks*) of a v-set in which every t-subset lies in exactly one block, and that a *large set* $\mathrm{LS}[N](t,\kappa,v)$ is a partition of all $\binom{v}{\kappa}$ many κ-subsets into N pairwise disjoint copies of $S(t,\kappa,v)$.

Theorem 2 (Reformulation)**.** *Let $k \ge 2$. The following are equivalent.*

(i) *The colouring asked for in Question 1 exists.*

(ii) $\chi(J(2k,k)) = k+1$.

(iii) *A large set* $\mathrm{LS}[k+1](k-1,k,2k)$ *exists.*

Theorem 2 is elementary, and the technique behind it is certainly known: the implication (iii) $\Rightarrow$ (ii) is the standard way of colouring a Johnson graph optimally, and Brouwer [6] records exactly the analogous statement for $J(n,3)$, deducing $\chi(J(n,3)) = n-2$ for $n \equiv 1,3 \pmod 6$, $n > 7$, from the existence of a large set of Steiner triple systems. What we use is the converse implication (ii) $\Rightarrow$ (iii), which relies on the coincidence $\binom{2k}{k+1} = \binom{2k}{k-1}$ special to $n = 2k$; the equivalence does not appear in [1], and it changes the complexion of the problem completely, because it makes the apparatus of design theory available. Our main application is the following.

Theorem 3. $\chi(J(20,10)) > 11$ *and* $\chi(J(24,12)) > 13$. *Equivalently, Question 1 has a negative answer for $k = 10$ and for $k = 12$.*

The proof, in Section 5, is three lines given Theorem 2: a colouring for k produces a Steiner system $S(k-1,k,2k)$, whose $(k-5)$-fold derived design is the Steiner quintuple system $S(4,5,k+5)$; and $S(4,5,15)$ and $S(4,5,17)$ are known not to exist [4].

Corollary 4. *The smallest open case of Question 1 is $k = 16$. A positive answer there would in particular produce a Steiner quintuple system $S(4,5,21)$, the smallest Steiner quintuple system whose existence is undecided [4, 5].*

Table 1 summarises what was known before and what is added here.

	previously known	this paper
$k=2$	works (classical)	—
$3 \le k \le 8$	fails (computer search)	reproduced from scratch; explained structurally
$k+1$ not prime	fails (Ma–Tang)	elementary reproof from divisibility
$k=10$	**open**	**fails** (Theorem 3)
$k=12$	**open**	**fails** (Theorem 3)
$k=16,18,22,\dots$	open	open; reduced to Conjecture 18
smallest open case	$k=10$	$k=16$
structure of the systems	—	Lemmas 12–14, Theorem 15

Table 1: Before and after.

The rest of the paper is organised as follows. Section 2 proves Theorem 2. Section 3 makes the shape of the claim explicit and works out the first three cases $k = 2,3,4$ in full. Section 4 derives an elementary reproof of the Ma–Tang theorem. Section 5 proves Theorem 3 and tabulates the resulting state of the problem. Section 6 develops the structure theory of tight systems and states Conjecture 18. Section 7 explains the previously known range $3 \le k \le 8$ in these terms and reports the verifying computations. Section 8 records Kramer–Mesner searches for $S(9, 10, 20)$ carried out before Theorem 3 was available; they are now logically redundant but corroborate it. Section 9 records a linear obstruction, and Section 10 lists open problems. All computations are reproducible; see Section 10.

2 The reformulation

We work on the ground set $[2k]$ and write $\binom{X}{j}$ for the family of j-subsets of X.

Lemma 5. *Let $k \ge 2$. Every edge of $J(2k, k)$ lies in a clique $\binom{A}{k}$ with $|A| = k + 1$, and every such $\binom{A}{k}$ is a clique with $k + 1$ members. Moreover $\omega(J(2k, k)) = k + 1$.*

Proof. If $|S \cap T| = k - 1$ then $|S \cup T| = k + 1$ and $S, T \in \binom{S \cup T}{k}$; conversely two distinct k-subsets of a $(k + 1)$-set meet in $k - 1$ points. For the clique number, let $\mathcal{C}$ be a clique with $|\mathcal{C}| \ge 3$ and $S \in \mathcal{C}$. Each $T \in \mathcal{C} \setminus \{S\}$ has the form $(S \setminus \{s_T\}) \cup \{x_T\}$ with $s_T \in S$, $x_T \notin S$. If some T_1, T_2 have $s_{T_1} \ne s_{T_2}$, then $|T_1 \cap T_2| = k - 1$ forces $x_{T_1} = x_{T_2} =: x$, and every member of $\mathcal{C}$ then lies in $\binom{S \cup \{x\}}{k}$, a set of size $k + 1$. Otherwise all $T \in \mathcal{C} \setminus \{S\}$ share a common $s_T = s$, and $\mathcal{C} \subseteq \{(S \setminus \{s\}) \cup \{x\} : x \notin S \setminus \{s\}\}$, of size $2k - (k - 1) = k + 1$. □

The two families in the proof are the two types of maximal clique of $J(2k, k)$: the *top cliques* $\binom{A}{k}$ with $|A| = k + 1$, and the *star cliques* $\{B \cup \{x\} : x \notin B\}$ with $|B| = k - 1$. Both have exactly $k + 1$ members, the second because $2k - (k - 1) = k + 1$. This coincidence, special to $n = 2k$, drives everything below.

Proof of Theorem 2. **(i) ⇔ (ii).** By Lemma 5 every edge lies in some top clique, so a colouring as in (i) is proper; conversely a proper $(k + 1)$-colouring must be rainbow on the $(k + 1)$-clique $\binom{A}{k}$.

(ii) ⇒ (iii). Let c be a proper $(k + 1)$-colouring with classes $C_0, \dots, C_k$. Fix i and count pairs (A, S) with $|A| = k + 1$, $S \in \binom{A}{k}$ and $c(S) = i$: each A contributes exactly one, and each such S lies in exactly $2k - k = k$ sets A. Hence

$$|C_i| \;=\; \frac{1}{k}\binom{2k}{k+1}. \tag{1}$$

Since C_i is independent, no two of its members share a $(k - 1)$-subset; each member contains k of them, so

$$k\,|C_i| \;\le\; \binom{2k}{k-1}. \tag{2}$$

But $\binom{2k}{k+1} = \binom{2k}{k-1}$, so (1) meets (2) with equality, which says precisely that every $(k-1)$-subset of $[2k]$ lies in exactly one member of C_i. Thus each C_i is a Steiner system $S(k - 1, k, 2k)$, and together they form a large set.

(iii) ⇒ (ii). Two blocks of a $S(k - 1, k, 2k)$ meeting in $k - 1$ points would share a $(k - 1)$-subset lying in two blocks; so each member of the large set is independent in $J(2k, k)$, and colouring by membership is proper. □

Remark 6. Equivalently $|C_i| = \binom{2k}{k}/(k + 1)$: a $(k + 1)$-colouring of $J(2k, k)$, if one exists, is automatically equitable.

3 What is being claimed, and the first three cases

It is worth being explicit about the logical shape of Question 1, because it is an *existence* question about k, not a statement about all k. It asks whether *some* $k > 2$ admits the colouring. The expected — and now largely confirmed — answer is that none does: $k = 2$ is the only case that works, and every $k > 2$ fails. What makes the problem hard is that the failure happens for three genuinely different reasons, and no single argument covers all of them.

(A) No Steiner system, for arithmetic reasons.
  $k + 1$ is not prime, the divisibility conditions (3) fail, and $S(k-1, k, 2k)$ does not exist. This is Corollary 8 and covers $k = 3, 5, 7, 8, 9, 11, 13, \ldots$

(B) No Steiner system, for design-theoretic reasons.
  $k + 1$ is prime, the arithmetic is fine, but the derived chain reaches a Steiner quintuple system that is known not to exist. This is Theorem 3 and covers $k = 10$ and $k = 12$.

(C) The system exists but the large set does not.
  A single $S(k-1, k, 2k)$ exists, and one still cannot find $k + 1$ pairwise disjoint copies. This is what happens at $k = 4$ and $k = 6$, and Conjecture 18 says it is what happens in every remaining case.

One remark on why the question is delicate rather than merely restrictive. Choosing a k-subset of a $(k+1)$-set means discarding exactly one of its $k+1$ elements, so a $(k+1)$-set has exactly $k+1$ many k-subsets — and there are exactly $k+1$ colours. So "all colours occur" is not a weak covering requirement but the assertion that those $k + 1$ subsets are *pairwise differently coloured*; the counts match with no slack at all. (Lemma 5 is this observation, stated as the clique structure of $J(2k, k)$.)

The first three cases exhibit exactly these three behaviours, so we record them explicitly.

$k = 2$: it works. The ground set is $\{1, 2, 3, 4\}$, there are $\binom{4}{2} = 6$ pairs, there are 3 colours, and each colour class must have $6/3 = 2$ members. A class must be an $S(1, 2, 4)$, i.e. a perfect matching of $\{1, 2, 3, 4\}$. There are exactly three of them and they are pairwise disjoint:

$$\{12, 34\}, \qquad \{13, 24\}, \qquad \{14, 23\}.$$

Together they use all six pairs. That is the large set, and the colouring.

$k = 3$: no system exists at all. The ground set is $\{1, \ldots, 6\}$, there are $\binom{6}{3} = 20$ triples and 4 colours, so a class would need $20/4 = 5$ members and would have to be a Steiner triple system $S(2, 3, 6)$: every pair of the six points in exactly one triple. Fix a point. It forms a pair with each of the other five, and every triple through it consumes two of those pairs, so it would lie in $5/2$ triples. Not an integer, so no $S(2, 3, 6)$ exists, and the question dies before the large set is even considered. (This is mechanism (A) in its smallest instance.)

$k = 4$: the system exists, the large set does not. The ground set is $\{1, \ldots, 8\}$, there are $\binom{8}{4} = 70$ quadruples and 5 colours, so each class needs $70/5 = 14$ members and must be a Steiner quadruple system SQS(8). These do exist — there are exactly 30 of them on a labelled 8-set, all isomorphic — and one can even find two disjoint ones:

$$\begin{array}{llllllllllllll} D_1: & 1234 & 1256 & 3456 & 1357 & 2457 & 2367 & 1467 & 2358 & 1458 & 1368 & 2468 & 1278 & 3478 & 5678 \\ D_2: & 1235 & 1346 & 2456 & 2347 & 1457 & 1267 & 3567 & 1248 & 3458 & 2368 & 1568 & 1378 & 2578 & 4678 \end{array}$$

D_1 and D_2 share no block, so together they account for 28 of the 70 quadruples. But an exhaustive check over all 30 systems shows that *no* SQS(8) is disjoint from both: the maximum number of pairwise disjoint systems is 2, while 5 are needed. This is mechanism (C), and it is the behaviour we conjecture to be universal.

4 Divisibility, and a reproof of the Ma–Tang theorem

For $S(t,\kappa,v)$ to exist it is necessary that

$$\binom{\kappa-i}{t-i} \,\Big|\, \binom{v-i}{t-i} \qquad (0 \le i \le t), \tag{3}$$

since the blocks through a fixed i-set form a $S(t-i,\kappa-i,v-i)$. Parameters satisfying (3) are called *admissible.*

Theorem 7. *Let $k \ge 2$ and $m = k+1$. The parameters $(k-1,k,2k)$ are admissible if and only if m is prime.*

Proof. With $t = k-1$, $\kappa = k$, $v = 2k$ one has $\binom{\kappa-i}{t-i} = k-i$, so (3) reads $(k-i) \mid \binom{2k-i}{k-1-i}$ for $0 \le i \le k-1$. Putting $j = k-i \in \{1,\dots,k\}$ this becomes $j \mid \binom{m+j-1}{m}$ for $1 \le j \le m-1$; and since

$$\frac{1}{j}\binom{m+j-1}{m} = \frac{(m+j-1)!}{m!\,j!} = \frac{1}{m+j}\binom{m+j}{j},$$

the condition is equivalent to

$$N \,\Big|\, \binom{N}{m} \qquad \text{for all } m < N < 2m. \tag{4}$$

If m is prime, then $\frac{m}{N}\binom{N}{m} = \binom{N-1}{m-1} \in \mathbb{Z}$ gives $N \mid m\binom{N}{m}$, hence $\frac{N}{\gcd(N,m)} \mid \binom{N}{m}$; and $\gcd(N,m) = 1$ because m is prime and $m < N < 2m$. So (4) holds.

If m is composite, let p be its least prime factor and $m = ps$ with $s \ge p \ge 2$. Put $N = m+p = p(s+1)$, legitimate since $p \le m/2 < m$. As $N - m = p$,

$$\binom{N}{m} = \binom{N}{p} = \frac{\prod_{i=0}^{p-1}(N-i)}{p!},$$

and for $1 \le i \le p-1$ we have $N - i \equiv -i \not\equiv 0 \pmod p$, so only the factor N contributes p-adically. Writing v_p for the p-adic valuation, $v_p\big(\prod_{i=0}^{p-1}(N-i)\big) = 1 + v_p(s+1)$ and $v_p(p!) = 1$, whence $v_p\big(\binom{N}{m}\big) = v_p(s+1) = v_p(N) - 1 < v_p(N)$, so $N \nmid \binom{N}{m}$. □

Corollary 8 (Ma–Tang, as reported in [1])**.** *If $k > 2$ and $k+1$ is not prime then $\chi(J(2k,k)) > k+1$.*

Proof. By Theorem 7 no $S(k-1,k,2k)$ exists, so no large set does; apply Theorem 2. □

We verified Theorem 7 numerically for all $k \le 400$; in each composite case the least failing N is exactly $m+p$, as in the proof.

5 Derived designs: the cases $k = 10$ and $k = 12$

Lemma 9 (Derived design)**.** *Let D be a $S(t,\kappa,v)$ on a point set X with $t \ge 1$, and let $x \in X$. Then $D_x = \{B \setminus \{x\} : x \in B \in D\}$ is a $S(t-1,\kappa-1,v-1)$ on $X \setminus \{x\}$. Consequently, if $S(t,\kappa,v)$ exists then so does $S(t-i,\kappa-i,v-i)$ for every $0 \le i \le t-1$.*

Proof. Let $T \subseteq X \setminus \{x\}$ with $|T| = t-1$. Then $T \cup \{x\}$ is a t-set, so it lies in exactly one block $B \in D$; that block contains x, so $B \setminus \{x\} \in D_x$ contains T. If $B' \setminus \{x\} \in D_x$ also contained T then B' would be a block containing $T \cup \{x\}$, so $B' = B$. Now iterate. □

For Question 1 at parameter k the relevant system is $S(k-1,k,2k)$, and by Lemma 9 its i-fold derived designs are $S(k-1-i,k-i,2k-i)$. Taking $i=k-5$ gives the member with $t=4$:

$$S(k-1,k,2k) \text{ exists} \quad\Longrightarrow\quad S(4,5,k+5) \text{ exists.} \tag{5}$$

Systems $S(4,5,v)$ are the *Steiner quintuple systems*. Their existence is a classical and delicate question: $S(4,5,11)$ is unique and exists, $S(4,5,15)$ and $S(4,5,17)$ do not exist, systems are known for $v \in \{23,35,47,71,83,107,131,167,243\}$, and the smallest v for which the question is open is $v=21$ [4, 5].

Proof of Theorem 3. Suppose Question 1 had a positive answer for $k=10$. By Theorem 2 there would be a large set LS[11](9, 10, 20); in particular $S(9,10,20)$ would exist. Five applications of Lemma 9 give

$$S(9,10,20) \to S(8,9,19) \to S(7,8,18) \to S(6,7,17) \to S(5,6,16) \to S(4,5,15),$$

contradicting the nonexistence of $S(4,5,15)$. For $k=12$, seven applications give

$$S(11,12,24) \to S(10,11,23) \to S(9,10,22) \to S(8,9,21) \to S(7,8,20) \to S(6,7,19) \to S(5,6,18) \to S(4,5,17)$$

contradicting the nonexistence of $S(4,5,17)$. □

Remark 10. For $k=10$ one may stop one step earlier, since $S(5,6,16)$ is also known not to exist; but $S(5,6,16)$ would itself yield $S(4,5,15)$, so the quintuple system is the fundamental obstruction.

Remark 11 (Provenance of the two nonexistence results). The nonexistence of $S(4,5,17)$ is the main theorem of [4]. The nonexistence of $S(4,5,15)$ is older and is normally quoted together with it (see [4] and the references therein, and [10, Part II]); the reader who wishes to make the present note self-contained should trace that attribution to its primary source. Only $k=10$ depends on it; $k=12$ rests on [4] alone.

Every member of the chain with $t \geq 5$ derives down to the $t=4$ member, and the members with $t \leq 3$ are Steiner quadruple and triple systems, whose existence is settled by congruence conditions already implied by admissibility. So (5) extracts all the information this method provides. Table 2 records the resulting state of the problem.

k	required system	$t=4$ member	that member	Question 1
2	$S(1,2,4)$	—	—	**positive**
4	$S(3,4,8)$	—	—	negative (no large set)
6	$S(5,6,12)$	$S(4,5,11)$	exists	negative (no large set)
10	$S(9,10,20)$	$S(4,5,15)$	**does not exist**	**negative**
12	$S(11,12,24)$	$S(4,5,17)$	**does not exist**	**negative**
16	$S(15,16,32)$	$S(4,5,21)$	unknown	**open**
18	$S(17,18,36)$	$S(4,5,23)$	exists	open
22	$S(21,22,44)$	$S(4,5,27)$	unknown	open
28	$S(27,28,56)$	$S(4,5,33)$	unknown	open
30	$S(29,30,60)$	$S(4,5,35)$	exists	open
36	$S(35,36,72)$	$S(4,5,41)$	unknown	open
40	$S(39,40,80)$	$S(4,5,45)$	unknown	open
42	$S(41,42,84)$	$S(4,5,47)$	exists	open

Table 2: Question 1 for the k with $k+1$ prime; every other $k>2$ is negative by Corollary 8. Bold entries are new.

6 The structure of tight Steiner systems

Call $S(k-1,k,2k)$ the *tight* parameters, since they meet the bound $v \geq (t+1)(\kappa-t+1)$ with equality. This section collects structural facts about such systems; they explain why the large set is so much harder to obtain than a single system, and they lead to Conjecture 18, which would settle Question 1 outright.

Throughout, $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the Catalan number; note $\frac{1}{k}\binom{2k}{k-1} = \frac{1}{k}\binom{2k}{k+1} = C_k$, so a tight system has exactly C_k blocks and $\binom{2k}{k} = (k+1)C_k$.

Lemma 12. *Let D be a $S(k-1,k,2k)$. Then every $(k+1)$-subset of $[2k]$ contains exactly one block of D.*

Proof. For $|A| = k+1$ put $a(A) = \#\{S \in D : S \subseteq A\}$. Counting incidences, $\sum_A a(A) = |D| \cdot k = \binom{2k}{k-1} = \binom{2k}{k+1}$, which is the number of such A. Counting ordered pairs, $\sum_A a(A)^2 = \sum_{S,S' \in D} \#\{A : S \cup S' \subseteq A\}$. The diagonal contributes $|D| \cdot k$. For $S \neq S'$ in D we have $|S \cap S'| \leq k-2$, since two blocks cannot share a $(k-1)$-set; hence $|S \cup S'| \geq k+2$ and no A contains both. So $\sum_A a(A)^2 = \sum_A a(A)$, i.e. $\sum_A a(A)(a(A)-1) = 0$. As the $a(A)$ are nonnegative integers, $a(A) \leq 1$ for all A, and since the $a(A)$ sum to the number of A's, $a(A) = 1$ throughout. $\square$

Corollary 13. *If D is a $S(k-1,k,2k)$ then so is $D^c := \{[2k] \setminus S : S \in D\}$. Moreover $\alpha(J(2k,k)) = C_k$ whenever a tight system exists, and the independent sets of that size are precisely the tight systems.*

Proof. A $(k-1)$-set B satisfies $B \subseteq [2k] \setminus S$ iff $S \subseteq [2k] \setminus B$, a $(k+1)$-set; apply Lemma 12. For the second claim, an independent set I has pairwise distinct shadows, so $k|I| \leq \binom{2k}{k-1}$, i.e. $|I| \leq C_k$, with equality iff every $(k-1)$-set is covered exactly once. $\square$

Thus tight systems are exactly the independent sets attaining the ratio (Hoffman) bound for $J(2k,k)$, whose least eigenvalue is $-k$ and whose valency is k^2: $\alpha \leq \binom{2k}{k} \cdot k/(k^2+k) = C_k$. Consequently, writing $\mathbf{1}$ for the all-ones vector and $V_0, \dots, V_k$ for the eigenspaces of the Johnson scheme,

$$\chi_D - \tfrac{1}{k+1}\mathbf{1} \;\in\; \ker \partial^{\downarrow} \cap \ker \partial^{\uparrow} = V_k, \tag{6}$$

where $\partial^{\downarrow}, \partial^{\uparrow}$ are the down and up operators of Section 9; indeed $\ker \partial^{\downarrow} = V_k$ is classical, and $\ker \partial^{\uparrow} = \Gamma(\ker \partial^{\downarrow}) = V_k$ where Γ is complementation. In particular $\dim(\ker \partial^{\downarrow} \cap \ker \partial^{\uparrow}) = \dim V_k = \binom{2k}{k} - \binom{2k}{k-1} = C_k$, which proves over $\mathbb{R}$ the statement we had verified over $\mathbb{F}_{k+1}$ for $k = 2,4,6$ in Section 9.

Lemma 14. *Γ acts on V_j as the scalar $(-1)^j$. Consequently, for k even every $S(k-1,k,2k)$ satisfies $D^c = D$, and for odd $k \geq 3$ no $S(k-1,k,2k)$ exists.*

Proof. The scalar action of Γ on V_j is standard (and we verified it numerically for $2k \leq 8$). By (6) and $\chi_{D^c} = \Gamma \chi_D$, $\chi_{D^c} - \frac{1}{k+1}\mathbf{1} = (-1)^k\big(\chi_D - \frac{1}{k+1}\mathbf{1}\big)$. For k even this gives $\chi_{D^c} = \chi_D$. For k odd it gives $\chi_{D^c} = \frac{2}{k+1}\mathbf{1} - \chi_D$, whose entries lie in $\{\frac{2}{k+1}, \frac{2}{k+1} - 1\}$; these are 0 or 1 only when $k = 1$. $\square$

The odd case is of course already covered by Corollary 8; the point is that the argument is independent of divisibility.

We now turn to disjointness. Let $D(k)$ denote the maximum number of pairwise disjoint $S(k-1,k,2k)$. A large set is exactly the statement $D(k) = k+1$.

Theorem 15. *Let D_1, D_2 be disjoint tight systems and $n_i = \#\{(S,T) \in D_1 \times D_2 : |S \cap T| = i\}$. Then, for every $0 \le j \le k$,*

$$n_{k-j} \;=\; \frac{C_k}{k+1}\binom{k}{j}\Big(\binom{k}{j} - (-1)^j\Big),$$

and moreover the same distribution is seen from every *block: for each $S \in D_1$,*

$$L_j := \#\{T \in D_2 : |S \cap T| = k-j\} = \frac{1}{k+1}\binom{k}{j}\Big(\binom{k}{j} - (-1)^j\Big),$$

independently of S. In particular each system is a completely regular code with respect to the other, $L_0 = 0$, $L_1 = k$, and for even k also $L_k = 0$.

Proof. Write $\chi_i = \frac{1}{k+1}\mathbf{1} + h_i$ with $h_i \in V_k$ by (6). Since $\langle \chi_1, \chi_2 \rangle = 0$ and $\langle \mathbf{1}, h_i \rangle = 0$, we get $\langle h_1, h_2 \rangle = -\frac{1}{(k+1)^2}\binom{2k}{k} = -\frac{C_k}{k+1}$. Let A_j be the association-scheme matrix for $|S \cap T| = k-j$, of valency $\binom{k}{j}^2$ (for $n = 2k$). Its eigenvalue on V_l is the Eberlein number $E_j(l) = \sum_i (-1)^i \binom{l}{i}\binom{k-l}{j-i}\binom{n-k-l}{j-i}$; at $l = k$ and $n = 2k$ only $i = j$ survives, so $E_j(k) = (-1)^j\binom{k}{j}$. Hence

$$n_{k-j} = \langle \chi_1, A_j \chi_2 \rangle = \tfrac{1}{(k+1)^2}\binom{2k}{k}\binom{k}{j}^2 + E_j(k)\langle h_1, h_2 \rangle,$$

which is the stated value. For the local statement, evaluate $A_j\chi_2 = \frac{1}{k+1}\binom{k}{j}^2\mathbf{1} + (-1)^j\binom{k}{j}h_2$ at a block $S \in D_1$, where $h_2(S) = -\frac{1}{k+1}$ because $S \notin D_2$. $\square$

We verified Theorem 15 against genuine disjoint pairs: for $k = 4$ the distribution is $(n_0, \ldots, n_4) = (0, 56, 84, 56, 0)$ with $L = (0, 4, 6, 4, 0)$, and for $k = 6$ it is $(0, 792, 3960, 7920, 3960, 792, 0)$ with $L = (0, 6, 30, 60, 30, 6, 0)$; in both cases the local counts are constant over all blocks, as predicted.

Remark 16. The numbers L_j are integers for every j if and only if $\binom{k}{j} \equiv (-1)^j \pmod{k+1}$ for every j, which holds if and only if $k+1$ is prime (for $q = k+1$ prime this is the classical $\binom{q-1}{j} \equiv (-1)^j$). We checked the equivalence for all $k \le 200$. So the Ma–Tang condition reappears here as exactly the integrality of the local intersection numbers of two disjoint tight systems.

Proposition 17. *$D(k) \ne k$. More precisely, if $D_1, \ldots, D_k$ are pairwise disjoint tight systems then the complement of their union is a further tight system, so that $D(k) = k+1$.*

Proof. Let $U = \bigsqcup_i D_i$ and $U' = \binom{[2k]}{k} \setminus U$. Each $(k-1)$-set has exactly one superset in each D_i, hence exactly k in U and exactly one in U'. Thus U' is a tight system. $\square$

Finally, the computation. For $k = 4$ and $k = 6$ we enumerated all tight systems and formed the *disjointness graph*, whose vertices are the systems and whose edges join disjoint pairs.

k	systems	degree	edges	triangles	$D(k)$
4	30	8	120	0	2
6	5,040	144	362,880	0	2

Table 3: The disjointness graph of tight Steiner systems is regular and triangle-free for $k = 4, 6$; hence at most two tight systems are pairwise disjoint, whereas a large set would need $k+1$.

So the large set fails not narrowly but by a wide margin: for $k = 6$ one needs seven pairwise disjoint systems and two is the maximum. (For $k = 2$, by contrast, the three perfect matchings of $[4]$ are pairwise disjoint and $D(2) = 3 = k+1$.) This suggests:

Conjecture 18. For every $k \geq 4$ no three tight Steiner systems $S(k-1, k, 2k)$ are pairwise disjoint; equivalently, the disjointness graph is triangle-free and $D(k) \leq 2$.

Proposition 19. *Conjecture 18 implies that Question 1 has a negative answer for every $k > 2$.*

Proof. For $k = 3$ no tight system exists. For $k \geq 4$ a large set needs $k + 1 \geq 5 > 2$ pairwise disjoint systems. □

Conjecture 18 is attractive because, unlike the approach of Section 5, it does not require knowing whether a single tight system exists — which for $k \geq 10$ is exactly the unresolved $t \geq 6$ question.

7 The known range $3 \leq k \leq 8$

Theorem 2 also explains the previously known range, splitting it into two different phenomena. For $k \in \{3, 5, 7, 8\}$ the number $k+1$ is composite, so no Steiner system exists at all (Theorem 7). For $k = 4$ and $k = 6$ the system does exist — $S(3, 4, 8) = \mathrm{SQS}(8)$ with 14 blocks, and the Witt/Mathieu system $S(5, 6, 12)$ with 132 blocks and automorphism group M_{12} [9] — and the obstruction is the large set.

We verified the latter two directly, with no colouring encoding involved. All SQS(8) are isomorphic, as are all $S(5, 6, 12)$, so the complete list of such systems on a labelled ground set is the S_{2k}-orbit of any one of them. We built one by exact cover, generated the orbit, and searched for $k + 1$ pairwise disjoint members by Algorithm X (Table 4); the count $5040 = 12!/|M_{12}|$ agrees with theory.

k	system	blocks	systems on $[2k]$	large set	search nodes
4	$S(3, 4, 8)$	14	30	does not exist	19
6	$S(5, 6, 12)$	132	5,040	does not exist	18,001

Table 4: Exhaustive verification that LS[5](3, 4, 8) and LS[7](5, 6, 12) do not exist.

As a cross-check we also solved Question 1 directly as a SAT instance for $2 \leq k \leq 8$ (Table 5), reproducing from scratch the range recorded in [1]. The case $k = 6$, about which Erdős and Rosenfeld were explicitly uncertain [1], took 61 seconds.

k	vertices	variables	clauses	result	time
2	6	18	75	satisfiable	< 0.1 s
3	20	80	564	unsatisfiable	< 0.1 s
4	70	350	3,855	unsatisfiable	< 0.1 s
5	252	1,512	24,198	unsatisfiable	< 0.1 s
6	924	6,468	142,303	unsatisfiable	61.2 s
7	3,432	27,456	796,232	unsatisfiable	0.2 s
8	12,870	115,830	4,285,719	unsatisfiable	77.5 s

Table 5: Direct SAT solution of Question 1. One clique is pinned to absorb the $(k+1)!$ colour symmetry.

8 Corroborating searches for $S(9, 10, 20)$

Before Theorem 3 was available we attacked $k = 10$ computationally, via the Kramer–Mesner method [8]: prescribe a group $G \leq S_{20}$ and search for a union of G-orbits on 10-sets covering each 9-set exactly once. Each search below is exhaustive for its group. The results are now subsumed by Theorem 3 — $S(9,10,20)$ does not exist at all — but they are an independent check on it, and the code carries over to $k = 16$.

prescribed group G	$\|G\|$	orbits (9-sets × 10-sets)	result
$\mathrm{PSL}(2,19)$	3,420	52×74	none
$\mathrm{AGL}(1,19)$	342	508×560	none
$F_{19} \rtimes C_9$	171	$988 \times 1{,}092$	none
$F_{19} \rtimes C_6$	114	$1{,}522 \times 1{,}676$	none
$F_{19} \rtimes C_3$	57	$2{,}960 \times 3{,}268$	none
C_{19}	19	$8{,}840 \times 9{,}724$	none
C_{20}	20	$8{,}398 \times 9{,}252$	none
D_{20}	40	$4{,}262 \times 4{,}752$	none
$C_{20} \rtimes \langle 3 \rangle$	80	$2{,}134 \times 2{,}368$	none

Table 6: Kramer–Mesner searches for $S(9,10,20)$ with prescribed automorphism group.

We also ran over cycle types of automorphisms of prime order: an automorphism of order p acts on 20 points as m p-cycles together with $20 - mp$ fixed points. Every cycle type was excluded for $p \in \{5,7,11,13,17,19\}$, as were $3^m \cdot 1^{20-3m}$ for $m \le 5$; the largest orbit system handled had $80{,}444 \times 87{,}516$ orbits.

The case $p = 11$ is instructive. Up to conjugacy S_{20} has a unique subgroup of order 11, generated by an 11-cycle with nine fixed points. No 10-subset is fixed by it, so all orbits on 10-sets have length 11; but the 9-set consisting of the nine fixed points is itself fixed, and its eleven 10-supersets form a single orbit, which therefore covers it eleven times. The orbit system is infeasible immediately.

9 A linear obstruction

Let $q = k+1$ be an odd prime and view a colouring as $c : \binom{[2k]}{k} \to \mathbb{Z}_q$. If c is valid then for each top clique the q values $c(A \setminus \{a\})$ exhaust $\mathbb{Z}_q$, so they sum to $q(q-1)/2 \equiv 0$; the same holds for star cliques. Hence c lies in

$$K = \ker \partial^{\uparrow} \cap \ker \partial^{\downarrow} \subseteq \mathbb{F}_q^{\binom{[2k]}{k}}, \qquad \partial^{\uparrow} c(A) = \sum_{S \in \binom{A}{k}} c(S), \quad \partial^{\downarrow} c(B) = \sum_{x \notin B} c(B \cup \{x\}),$$

for $|A| = k+1$ and $|B| = k-1$. Over $\mathbb{R}$ this space is exactly the top eigenspace V_k of the Johnson scheme, of dimension $\binom{2k}{k} - \binom{2k}{k-1} = C_k$, as recorded at (6). Exact rank computations over $\mathbb{F}_q$ give $\dim K = 2, 14, 132$ for $k = 2, 4, 6$, with both maps of full row rank — the same numbers.

Conjecture 20. For $k+1$ an odd prime, $\dim_{\mathbb{F}_{k+1}} K = C_k$, i.e. the modular rank agrees with the characteristic-zero rank.

10 Open problems

(1) $k = 16$ **and** $S(4,5,21)$**.** By Corollary 4 the smallest open case of Question 1 needs $S(4,5,21)$, whose existence is a long-standing open problem and whose possible automorphism groups have been narrowed considerably [5]. A nonexistence proof for $S(4,5,21)$ would settle $k = 16$. The converse fails: the chain condition is only necessary.

(2) **Conjecture 18.** Prove that no three tight systems are pairwise disjoint for $k \geq 4$. This is the one route we know of that would settle Question 1 in full, and it bypasses the $t \geq 6$ existence question entirely. Two natural handles: the local structure of Theorem 15, and the observation that for S in one system and any other system D_i the induced map $a \mapsto \varphi_i(S \setminus \{a\})$ is a bijection $S \to [2k] \setminus S$, so m pairwise disjoint systems produce, at each block, $m-1$ pairwise discordant bijections — a partial Latin square of order k.

(3) **Beyond the $t = 4$ member.** For $k \in \{18, 30, 42\}$ the quintuple system $S(4, 5, k+5)$ exists, so (5) gives nothing. Is there a necessary condition coming from the *large set* rather than from a single system that handles these? For $k = 4$ and $k = 6$ it is precisely the large set that fails.

(4) **Does $S(k-1, k, 2k)$ ever exist for $k > 6$?** Any such system has $t = k-1 \geq 9$, and no nontrivial Steiner system with $t \geq 6$ has ever been constructed; Keevash's existence theorem [7] is non-constructive and requires v large relative to t and κ, whereas here $v = 2k$ is as small as the bound $v \geq (t+1)(\kappa - t + 1)$ permits. A negative answer for every $k > 6$ with $k+1$ prime would settle Question 1 completely.

(5) **Prove Conjecture 20**, presumably by decomposing the permutation module $\mathbb{F}_q^{\binom{[2k]}{k}}$ for S_{2k} in characteristic $q = k+1$.

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Code and data availability

The following Python programs (using `python-sat` with the CaDiCaL backend) reproduce every computational claim above.

file	contents
`erdos835.py`	exact and C_q-equivariant SAT encodings; independent verifier
`steiner_check.py`	class size vs. shadow bound; admissibility table
`steiner_check2.py`	admissibility $\Leftrightarrow$ $k+1$ prime; small Steiner systems
`largeset2.py`	enumeration of all systems; Algorithm X large-set search
`kramer_mesner.py`	prescribed-group searches for $S(k-1, k, 2k)$
`prime_autos.py`	prime-order automorphism sweep
`ranks.py`	ranks of $\partial^{\uparrow}, \partial^{\downarrow}$ over $\mathbb{F}_q$
`verify_chain.py`	derived-design chains and the status table
`structure.py`	Lemmas 12–14; Γ on V_j; $D(k)$ by maximum clique
`pairdist.py`	Theorem 15 checked against real disjoint pairs
`localdist.py`	local numbers L_j; the primality equivalence of Remark 16
`disjgraph.py`	degree and triangle count of the disjointness graph

Every solver output is re-checked by an independent verifier enumerating all $(k+1)$-subsets directly. The Kramer–Mesner code was validated by rediscovering the Mathieu system $S(5, 6, 12)$ from $\mathrm{PSL}(2, 11)$, and the SAT encoding by reproducing the known answers for $2 \leq k \leq 8$.

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